MOMENT DISTRIBUTION METHOD

To simplify the derivation, one assumes:

(a) Prismatic member (constant I)

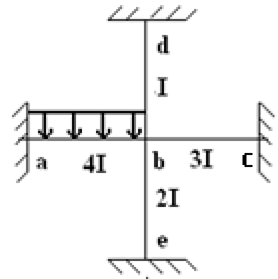
(b) No member chord rotation ($\Psi = 0$)

(c) No joint translation

(d) No support settlements

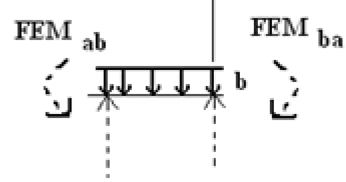
Step-by-Step Procedures

Step 1: Consider an unload structure and joint b is also fixed (locked)

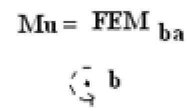


Step 2: When the uniform load acts on the structure,
FEM_{ab} and FEM_{ba} are developed and can be computed.

Step 3: This FEM will cause joint b to rotate CCW,
when joint b is unlocked => unbalanced moment



FBD of member ab

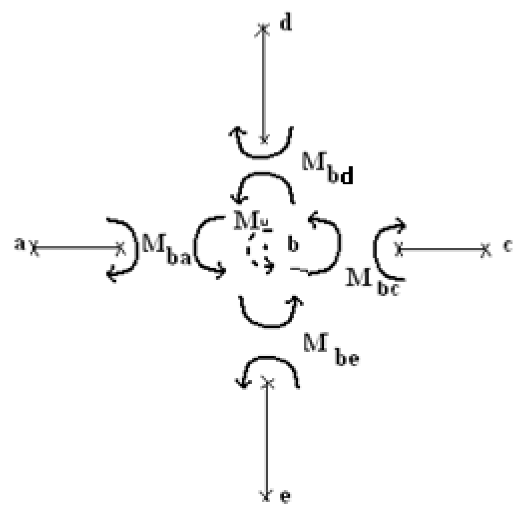


FBD of Joint b

Step 4: When joint b rotates, certain moments are developed through out the length of all members meeting at joint b
=> distributed moments

Step 5: Joint b will continue to rotate until sufficient distributed moments are developed at the "b" ends of these members to counter balance the effect of M_u.

Step 6: Simultaneously, end moments are developed at the far ends of these members
=> carry over moments



Step 7: When equilibrium at joint b is established, the total end moments of various members will be the algebraic sum of the FEM and the moments caused by the rotation of joint b.

$$\sum M_{\text{at Joint b}} = 0 \;\circlearrowleft\!+ \;=\; M_u + \underbrace{M_{ba} + M_{bc} + M_{bd} + M_{be}}_{\text{Distributed moments}} \qquad (100)$$

Using Slope Deflection Equation (SDE), with

$\theta_a = 0 = \theta_c = \theta_d = \theta_e$ (Because fixed supports)

$\Psi_{ba} = 0 = \Psi_{bc} = \Psi_{bd} = \Psi_{be}$ (Because no members' chord rotations)

Zero, because it has already been accounted in the unbalanced moment M_u

One obtains:

$$M_{ba} = 2E\left(K_{ba} = \frac{I}{L}\right)\left[2\theta_b + \cancel{\theta_a} - \left(3\cancel{\Psi_{ba}} = \frac{\Delta}{L}\right)\right] + \cancel{FEM_{ba}} \qquad (101)$$

zero

$$\left.\begin{aligned} M_{ba} &= 4EK_{ba}\theta_b \\ \text{Similarly, } M_{bc} &= 4EK_{bc}\theta_b \\ M_{bc} &= 4EK_{bd}\theta_b \\ M_{be} &= 4EK_{bc}\theta_b \end{aligned}\right\} \qquad (102)$$

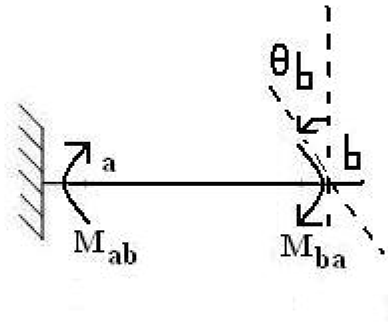
Substituting Eq. (102) into Eq. (100): $4E\theta_b(\sum_b K \equiv K_{ba} + K_{bc} + K_{bd} + K_{be}) + M_u = 0$

Thus: $\theta_b = \dfrac{-M_u}{4E(\sum K)}$ (103)

Substituting Eq. (103) into Eq. (102):

$$\left.\begin{aligned} M_{ba} &= \cancel{4E}K_{ba} * \frac{-M_u}{\cancel{4E}(\sum K)} = -\left(DF_{ba} \equiv \frac{K_{ba}}{\sum K}\right) * M_u \\ M_{bc} &= -\left(DF_{bc} \equiv \frac{K_{bc}}{\sum K}\right) * M_u \end{aligned}\right\} \qquad (104)$$

How to compute carry-over-moment ($\mathbf{M_{ab}}$) from distribute moment ($\mathbf{M_{ba}}$)?



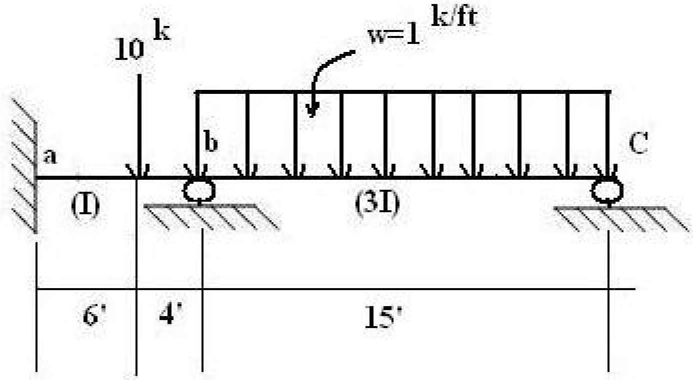
From Eq. (101), one has:

$$M_{ba} = 2E(K_{ba})[2\theta_b]$$

$$M_{ab} = 2E(K_{ba})[\underbrace{\cancel{2\theta_a}}_{\text{Zero}} + \theta_b]$$

$$\Longrightarrow \quad M_{ab} \equiv \text{Carry} - \text{over} - \text{moment} = \frac{1}{2} M_{ba} \qquad (105)$$

Example 1:



$$K_{ba} = K_{ab} = \frac{I}{10} = 0.1\ I$$

$$K_{bc} = K_{cb} = \frac{3I}{15} = 0.2\ I$$

$$DF_{ba} = \frac{K_{ba}}{\sum_b K} = \frac{0.1\ I}{(0.1\ I + 0.2\ I)} = \frac{1}{3}$$

$$DF_{ab} = \frac{0.1\ I}{(\sum_a K = 0.1\ I)} = 1$$

$$DF_{bc} = \frac{K_{bc}}{\sum_b K} = \frac{0.2\ I}{0.3\ I} = \frac{2}{3} \qquad DF_{cb} = \frac{0.2\ I}{(\sum_c K = 0.2\ I)} = 1$$

	Joint a (Fixed)	Joint b (Roller)	Joint b (Roller)	Joint c (End roller)
Members	ab	ba	bc	cb
D.F.	1	$\frac{1}{3}$	$\frac{2}{3}$	1
FEM (+)	-9.6 K/ft	+14.4	-18.75	+18.75 (unbalanced moment at joint C)
Unlock jt. C		unbalanced moment at joint b	-9.38 carry over	-18.75 (distributed moment M_{cb} is developped)
Unlock jt. B	+2.29 carry over	+4.58 (distributed moments M_{ba} and M_{bc} are developped)	+9.15 carry over →	+4.58
Unlock jt. C			-2.29	-4.58
Unlock jt. B	+0.38	+0.76	+1.53	+0.76
Unlock jt. C			-0.38	-0.76
Unlock jt. B	+0.07	+0.13	+0.25	+0.12
	$\overline{M_{ab}} = -6.85$	$\overline{M_{ba}} = 19.89$	$\overline{M_{bc}} = -19.89$	$\overline{M_{cb}} \approx 0$

Reduced Member Stiffness with End Roller (or Pin) Support

- In Example1, convergence is rather slow because the (end) joint C is actually a "hinge end", and continuously throws back a sizable carry-over-moments to joint b.
- Also, joint c should be unlocked and remains to be unlocked (as a hinge support), so that joint C will not be continuously received the carry-over-moment from joint b.
- Using SDE, with joint C is consider as a hinge support (not fixed support, hence $\theta_C \neq 0$) one obtains:

$M_{cb} = 2E(K_{Cb})[2\theta_C + \theta_b] = 0$ (Since moment at the hinge joint C is zero)
Hence:

$$\theta_C = -\frac{\theta_b}{2}$$

Also, $M_{bc} = 2E(K_{bc})[2\theta_b + \theta_C] = 2E(K_{bC})\left[2\theta_b - \frac{\theta_b}{2}\right] = 2E(K_{bC})\left[\frac{3\theta_b}{2}\right]$

Or: $M_{bC} = E(K_{bC})[3\theta_b] = 4E\left(\frac{3}{4}K_{bC}\right)[\theta_b] = 4E\left(K_{bC}^R\right)[\theta_b]$ ________________ (108)

Comparing Eq. (108) and Eq. (102), they look "very similar". The only difference is that K_{bc} (in Eq. 102) is replaced by the "reduced stiffness"

$K_{bC}^R \equiv \frac{3}{4}K_{bC}$ ____ (109)

Example 2: Re-do Example 1 (but using reduced stiffness K_{bc}^{R})

$$K_{ba} = K_{ab} = \frac{I}{10}$$

$$K_{bc}^{R} = \left(\frac{3}{4}\right) K_{bc} = \left(\frac{3}{4}\right)\left(\frac{3I}{15}\right) = \frac{9I}{60} = \frac{3I}{20}$$

$$\sum_{b} K = K_{ba} + K_{bc}^{R} = \frac{I}{10} + \frac{3I}{20} = \frac{5I}{20}$$

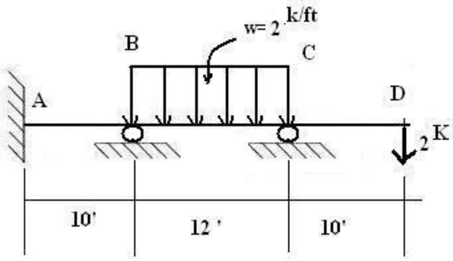
$$\rightarrow DF_{ba} = \frac{K_{ba}}{\sum_b K} = \frac{I}{10} * \frac{20}{5I} = 0.4$$

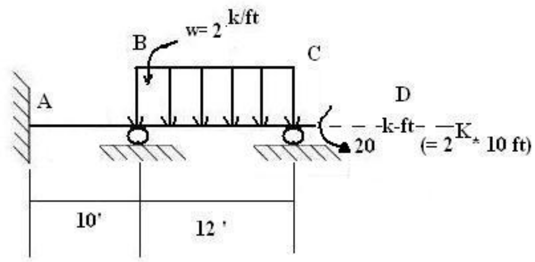
$$\rightarrow DF_{bc} = \frac{K_{bc}^{R}}{\sum_b K} = \frac{3I}{20} * \frac{20}{5I} = 0.6$$

	Joint a (Fixed)	Joint b (roller)	Joint b (roller)	Joint c (end roller)
Member	ab	ba	bc	cb
DF	1	0.4	0.6	1
FEM	-9.6^{Kft}	+14.4	-18.75	+18.75 (unbalanced moment at Joint C)
Unlock jt. C			-9.38 ← $\frac{\text{carry}}{\text{over}}$	-18.75
Unlock jt.b	+2.75← $\frac{\text{carry}}{\text{over}}$	+5.49	+8.24	N/A
	$\overline{\overline{M_{ab}}} = -6.85$	$\overline{\overline{M_{ba}}} = 19.89$	$\overline{\overline{M_{bc}}} = -19.89$	$\overline{\overline{M_{cb}}} = 0$

(Unbalanced moment at Joint b)

<u>Example 3</u>: (over hanging beam)





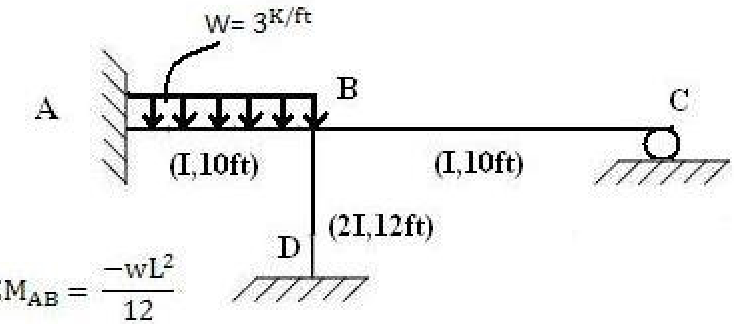
$$DF_{BA} = \frac{K_{BA}}{\sum_B K} = \frac{I}{10} * \frac{80}{13I} = \frac{8}{13}$$

$$DF_{BC} = \frac{K_{BC}^R}{\sum_B K} = (\frac{3}{4} * \frac{I}{12}) * \frac{80}{13I} = \frac{5}{13}$$

$$\sum_B K = K_{BA} + K_{BC}^R = \left(\frac{I}{10}\right) + \left(\frac{3I}{48} = \frac{I}{16}\right) = \frac{13I}{80}$$

	Joint A (fixed)	Joint B (roller)	Joint B (roller)	Joint C (end roller)	
Member	AB	BA	BC	CB	CD
DF	1	$\frac{8}{13}$	$\frac{5}{13}$	1	
FEM	0^{K-ft}	0^{Kft}	-24	+24	-20^{Kft}
Unlock jt. C			-2 ←	-4	
Unlock jt. B	+8	← +16	+10 - - - →	N/A	
	$\overline{M_{AB}} = 8$	$\overline{M_{BA}} = 16$	$\overline{M_{BC}} = -16$	$\overline{M_{CB}} = 20$	$\overline{M_{CD}} = -20$

Example 4: (Frame Structure)



$$FEM_{AB} = \frac{-wL^2}{12}$$

$$FEM_{AB} = -25^{Kft}$$

$$K_{BA} = \frac{I}{10} \rightarrow DF_{BA} = \left(\frac{I}{10}\right)\left(\frac{120}{41\ I}\right) = \frac{12}{41}$$

$$K_{BC}^{R} = \left(\frac{3}{4}\right)\left(\frac{I}{10}\right) = \frac{3I}{40}$$

$$\rightarrow DF_{BC} = \left(\frac{3I}{40}\right)\left(\frac{120}{41\ I}\right) = \frac{9}{41}$$

$$K_{BD} = \frac{2I}{12} = \frac{I}{6}$$

$$\rightarrow DF_{BD} = \qquad\qquad = \frac{20}{41}$$

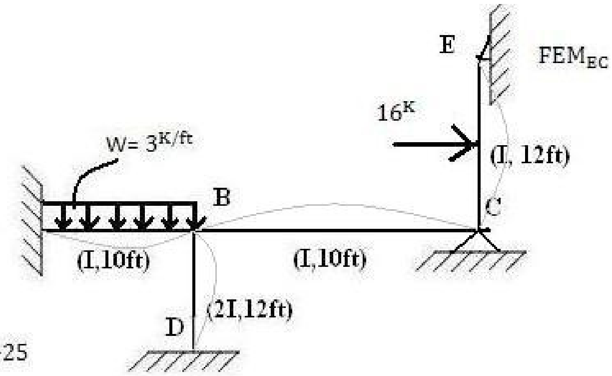
$$\sum_{B} K = \left(\frac{I}{10} + \frac{3I}{40} + \frac{I}{6}\right) = \frac{12I + 9I + 20I}{120} = \frac{41I}{120}$$

	Joint A (fixed)	Joint B (free)			Joint C (End Roller)	Joint D(fixed)
Member	AB	BA	BC	BD	CB	DB
DF	1	12/41	9/41	20/41	1	1
FEM	-25^{kjt}	+25	0	0	0	0
Unlock jt. B	-3.66 ←	-7.32	-5.49	-12.2	N/A	-6.1

$\overline{M_{AB} = -28.66}$ $\overline{M_{BC} = -5.49}$

$\overline{M_{BA} = 17.68}$ $\overline{M_{BD} = -12.2}$ $\overline{M_{CB} = 0}$ $\overline{M_{DB} = -6.1}$

<u>Example 5:</u> (Frame Structure)



$$K_{BA} = \frac{I}{10} \rightarrow DF_{BA} = \frac{K_{BA}}{\sum K_B} = \frac{3}{11}$$

$$K_{BD} = \frac{2I}{12} = \frac{I}{6} \rightarrow DF_{BD} = \frac{K_{BD}}{\sum K_B} = \frac{5}{11}$$

$$K_{BC} = \frac{I}{10} \rightarrow DF_{BC} = \frac{3}{11}$$

$$\sum K_B = K_{BA} + K_{BD} + K_{BC} = \frac{11I}{30}$$

$$FEM_{AB} = \frac{-wL^2}{12} = -25$$

$$K_{CB} = \frac{I}{10} \rightarrow DF_{CB} = \left(\frac{I}{10}\right)\left(\frac{80}{13I}\right) = \frac{8}{13}$$

$$K_{CE}^R = \left(\frac{3}{4}\right)\left(\frac{I}{12}\right) = \frac{I}{16} \rightarrow DF_{CE} = \frac{5}{13}$$

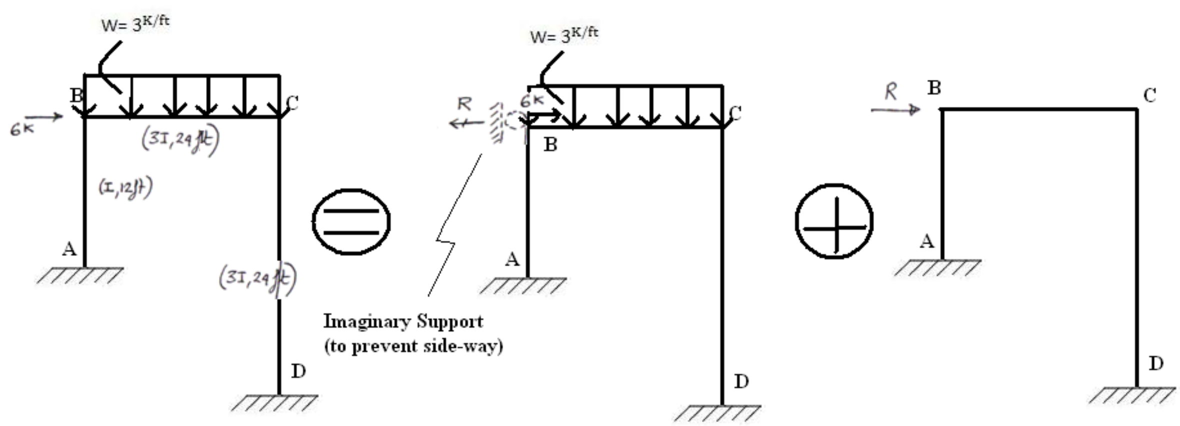
$$\sum K_C = K_{CB} + K_{CE}^R = \frac{13I}{80}$$

(COM = Carry Over Moment)

	Jt. A (Fixed)	Jt. B (Free)			Jt. C (pin)		Jt. D (Fixed)	Jt. E (end pin)
Member	AB	BA	BD	BC	CB	CE	DB	EC
DF	1	3/11	5/11	3/11	8/13	5/13	1	1
FEM	-25^{Kft}	+25	0	0	0	-24	0	+24
Unlock jt. E								-24
COM						-12		
Unlock jt. C					+22.15	+13.85		
COM				+11.08				
Unlock jt. B		-9.84	-16.4	-9.84				
COM	-4.92				-4.92		-8.2	
Unlock jt. C					+3.03	1.89		
COM				1.52				
Unlock jt. B		-0.41	-0.69	-0.41				
COM	-0.20				-0.20		-0.34	
Unlock jt. C					+0.12	+0.08		
COM				0.06				
	$M_{AB} = -30.12$	$M_{BA} = 14.75$	$M_{BD} = -17.09$	$M_{BC} = 2.41$	$M_{CB} = 20.18$	$M_{CE} = -20.18$	$M_{DB} = -8.54$	$M_{EC} = 0$

Moment Distribution (with Sideway)

Example 6:



Case A: No side-way **Case B**: with side-way

$$[FEM_{BC} = \frac{-wL^2}{12} = -144^{Kft}$$
$$FEM_{CB} = +144]$$

$$\begin{cases} K_{BA} = \frac{I}{12} \rightarrow DF_{BA} = \left(\frac{I}{12}\right)\left(\frac{24}{5I}\right) = 0.4 \\ K_{BC} = \frac{3I}{24} = \frac{I}{8} \rightarrow DF_{BC} \qquad = 0.6 \\ \sum K_B = \frac{I}{12} + \frac{I}{8} = \frac{5I}{24} \end{cases}$$

$$\begin{cases} K_{CB} = \frac{3I}{24} = \frac{I}{8} \rightarrow DF_{CB} = \frac{K_{CB}}{\sum K_C} = 0.5 \\ K_{CD} = \frac{3I}{24} = \frac{I}{8} \rightarrow DF_{CD} = \qquad = 0.5 \\ \sum K_C = \frac{I}{8} + \frac{I}{8} = \frac{I}{4} \end{cases}$$

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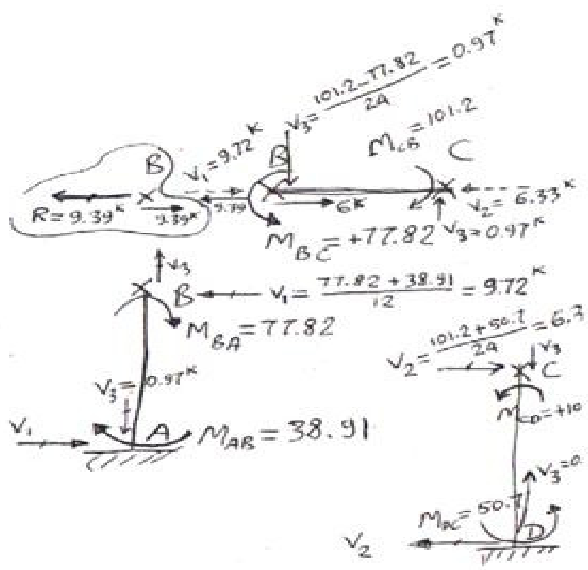
Case A:

	Jt. A (Fixed)	Jt. B (Imaginary roller)		Jt. C (Free)		Jt. D (Fixed)
Member	AB	BA	BC	CB	CD	DC
DF	1	0.4	0.6	0.5	0.5	1
FEM	0^{Kft}	0	144^{Kft}	144	0	0
Unlock jt B		+57.6	+86.4			
COM	28.8			43.2		
Unlock jt C				-93.6	-93.6	
COM			-46.8			-46.8
Unlock jt B		+18.72	+28.8			
COM	9.36			14.04		
Unlock jt C				-7.02	-7.02	
COM			-3.51			-3.51
Unlock jt B		+1.40	+2.10			
COM	0.70			1.05		
Unlock jt C				-0.52	-0.52	
COM			-0.26			-0.26
Unlock jt B		+0.10	+0.16			
COM	0.05			0.08		
Unlock jt C				-0.04	-0.04	
COM			-0.02			-0.02
	$M_{AB} = 38.91$	$M_{BA} = 77.82$	$M_{BC} = -77.82$	$M_{CB} = 101.2$	$M_{CD} = -101.2$	$M_{DC} = -50.7$



$$\overset{+}{\rightarrow} \quad \Sigma F_x = 0 = V_1 - V_2 + 6^K - R$$

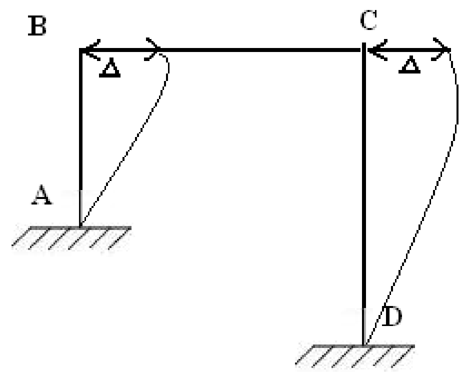
$$0 = 9.72 - 6.33 + 6 - R$$

So: $R = +9.39^K$

<u>Case B:</u> Fixed all joints, and applied side-way

$$\begin{cases} M_{AB} = 2E\left(\frac{I}{12}\right)\left[2\cancel{\theta_A} + \cancel{\theta_B} - 3\left(\Psi_{AB} = \frac{+\Delta}{12}\right)\right] + \cancel{FEM_{AB}} \\ M_{AB} = \frac{-EI\Delta}{24} = M_{BA} \end{cases}$$

$$\begin{cases} M_{CD} = 2E\left(\frac{3I}{24}\right)\left[2\cancel{\theta_C} + \cancel{\theta_D} - 3\left(\Psi_{CD} = \frac{+\Delta}{24}\right)\right] + \cancel{FEM_{CD}} \\ M_{CD} = \frac{-3EI\Delta}{96} = M_{DC} \end{cases}$$



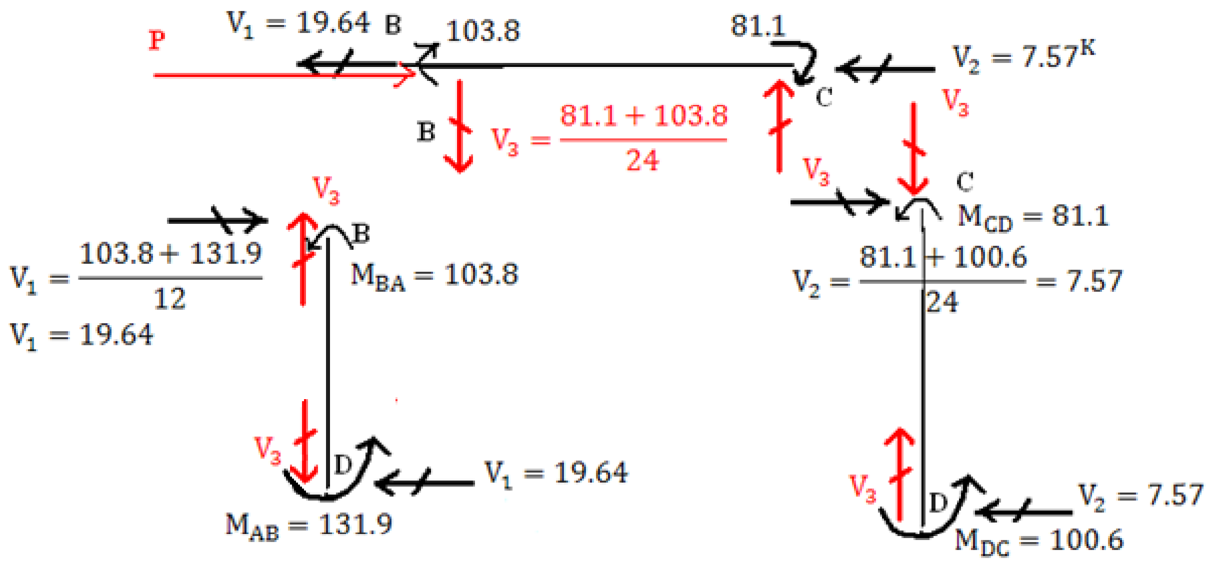
Now, arbitrarily select $EI\Delta \equiv 3840$

Hence, $M_{AB} = -160 = M_{BA} \equiv FEM_{\text{Due to settlement }\Delta}$
$M_{CD} = -120 = M_{DC} \equiv FEM_{\text{Due to settlement }\Delta}$

	Jt. A (Fixed)	Jt. B (Free)		Jt. C (Free)		Jt. D (Fixed)
Member	AB	BA	BC	CB	CD	DC
DF	1	0.4	0.6	0.5	0.5	1
FEM	-160^{Kft}	-160	0	0	-120	-120
Unlock B	+32	+64	+96	+48		
Unlock C			+18	+36	+36	+18
Unlock B	-3.6	-7.2	-10.8	-5.4		
Unlock C			1.35	+2.7	+2.7	1.35
Unlock B	-0.27	-0.54	-0.81	-0.40		
Unlock C			0.10	+0.20	0.20	0.10
Unlock B	-0.02	-0.04	-0.06	-0.03		
	$M_{AB} = -131.9$	$M_{BA} = -103.8$	$M_{BC} = 103.8$	$M_{CB} = 81.1$	$M_{CD} = -81.1$	$M_{DC} = -100.6$

Determine the force P which causes
$EI\Delta = 3840$

From Case B:



$$(\sum F_x)_{\text{Member BC}} = 0 \overset{+\rightarrow}{=} P - V_1 - V_2$$
$$0 = P - 19.64 - 7.57$$
$$\text{Hence: } P = +27.21^K$$

However, from case A, one only has $R = 9.39^K$

Thus, case B results have to be scaled down by a ratio Ratio $= \frac{9.39^K}{27.21^K} = 0.345$

New results for case B:

$$(M^*_{AB})_{\text{case B}} = (M_{AB})_{\text{case B}} * 0.345 = (-131.9)(0.345) = -45.52$$
$$(M^*_{BA})_{\text{case B}} = (-103.8)(0.345) = -35.81$$
$$(M^*_{BC})_{\text{case B}} = (+103.8)(0.345) = +35.81$$
$$(M^*_{CB})_{\text{case B}} = (81.1)(0.345) = 27.98$$
$$(M^*_{CD})_{\text{case B}} = (-81.1)(0.345) = -27.98$$
$$(M^*_{DC})_{\text{case B}} = (-100.6)(0.345) = -34.71$$

Superposition of Case A and Case B

The final end-moments of the original problem can be given as:

$$M_{AB} = (M_{AB})_{case\ A} + (M^*_{AB})_{case\ B} = (38.91) + (-45.52) = -6.61^{K-ft}$$
$$M_{BA} = (M_{BA})_{case\ A} + (M^*_{BA})_{case\ B} = (77.82) + (-35.81) = 42$$
$$M_{BC} = (M_{BC})_{case\ A} + (M^*_{BC})_{case\ B} = (-77.82) + (+35.81) = -42$$
$$M_{CB} = \quad = (101.2) + (27.98) = 129.1$$
$$M_{CD} = \quad = (-101.2) + (-27.98) = -129.1$$
$$M_{DC} = \quad = (-50.7) + (-34.71) = -85.4$$